**Decidability Exercise Sheet**

For each problem indicate decidability and explain.

1. *Given ™ T does it ever reach a state other than initial if starting with blank input tape*

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| This problem is decidable. As T proceeds in its computation, one of the following will occur:   1. T changes state. 2. T crashes without having changed its state. 3. T moves its tape head to the right without having changed state. 4. T writes a symbol on the first square (0)   In a and b the problem is decidable. For c and d, T must be an infinite loop, thus having constraints placed upon them. |

1. *Given ™ T and a non-halting state q of T, does T ever enter state q when it begins with blank tape.*

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| This problem is undecidable. To show this is true, we reduce the problem of whether the Turing Machine accepts the empty string(undecidable). Given ™ T we construct ™ T’ which has all states in T plus additional one called q. T’ works by making exactly the same moves as T, except that if T ever halts, T’ moves instead to state q, then halts on the next move. T accepts the empty string if and only if T’ enters state q. |

1. *Given ™ T and non-halting state q of T, is there an input string x that would cause T to eventually enter state q.*

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| The problem is undecidable. We can use the same construct above (T and T’) to prove this case. |

1. *Given ™ T, does T eventually enter every one of its non halting states if it begins with blank tape.*

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| This problem is undecidable. We set up the same solution as before although all the symbols of T are included plus an additional one, $. For any move that would cause T to halt, instead T’ moves to start state of T and places $ on the tape. Thereafter the $ causes T’ to cycle through all its non- halting states, q being last.  If T accepts the empty string then some moves cause T to halt and therefore T’ enters all of its non-halting states. If T does not accept the empty string, then since T never executes a halting move after starting with a blank tape, T’ will never enter state q and so does not enter all of its non halting states. |

1. *Given ™ T, is there an input string that causes T to enter every one of its non-halting states*

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| This problem is undecidable. We can use exactly the same construction in q4 to show that this is the case. |

1. *Given ™ T, does it accept the empty string in an even number of moves.*

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| The problem is undecidable. We reduce the halting problem to it. Given ™ T, We construct a ™ T’ which copies T but also keeps track of whether it has made an even number of moves or an odd number. The other difference in T and T’ is that if T halts after an odd number moves then T’ makes an additional move before halting. The result produced is that T’ accepts exactly the same strings as T, but halts only after an even number of moves. Therefore, T halts if and only if T’ halts after an even number of moves. |

1. *Given ™ T, is there a string it accepts in an even number of moves*

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| This problem is undecidable. We can use q6 as a basis for this proof. |

1. *Given ™ T, and string w, does T loop forever on input w*

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| This problem is undecidable. To prove this, we reduce that ™ T halts over a given string. A ™ T’ is constrcuted that accepts the same language but never crashes. Then for any input string w, T fails to accept w if and only if T’ loops forever on w. This means that the problem of whether or not a given Turing machine halts over a given string is reducible to this one. |

1. *Given ™ T, are there any input strings on which T loops forever.*

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| This problem is undecidable. We can use q8 as a base for this problem |

1. *Given ™ T, does T halt within 10 moves on every string.*

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| This problem is decidable! The reason is that within 10 moves, a ™ cannot move its tape head any further right than square 10. Therefore, we can look at the first 10 moves T makes for every possible input string of length 10 or less. At the end of this process, we will know whether T halts within 10 moves on every string. |